5 The Solow Growth Model

5.1 Models and Assumptions

- What is a model? A mathematical description of the economy.

- Why do we need a model? The world is too complex to describe it in every detail.

- What makes a model successful? When it is simple but effective in describing and predicting how the world works.

- A model relies on simplifying assumptions. These assumptions drive the conclusions of the model. When analyzing a model it is crucial to spell out the assumptions underlying the model.

- Realism may not a the property of a good assumption.

5.2 Basic Assumptions of the Solow Model

1. Continuous time.

2. Single good produced with a constant technology.

3. No government or international trade.

4. All factors of production are fully employed.

5. Labor force grows at constant rate $n = \frac{\dot{L}}{L}$.

6. Initial values for capital, K_0 and labor, L_0 given.

Production Function

- Neoclassical (Cobb-Douglas) aggregate production function:

$$Y(t) = F[K(t), L(t)] = K(t)^{\alpha} \; L(t)^{1-\alpha}$$

- To save on notation write: $Y = A\; K^{\alpha}\; L^{1-\alpha}$

- Constant returns to scale:
  $F(\lambda K, \lambda L) = \lambda\; F(K, L) = \lambda A\; K^{\alpha}\; L^{1-\alpha}$

- Inputs are essential: $F(0, 0) = F(K, 0) = F(0, L) = 0$

- Marginal productivities are positive:

$$\begin{aligned}\frac{\partial F}{\partial K} &= \alpha A K^{\alpha-1} L^{1-\alpha} > 0 \\ \frac{\partial F}{\partial L} &= (1-\alpha)\, A K^{\alpha} L^{-\alpha} > 0\end{aligned}$$

- Marginal productivities are decreasing,

$$\begin{aligned}\frac{\partial^2 F}{\partial K^2} &= (\alpha-1)\, \alpha A\, K^{\alpha-2} L^{1-\alpha} < 0 \\ \frac{\partial^2 F}{\partial L^2} &= -\alpha\, (1-\alpha)\, A\, K^{\alpha} L^{-\alpha-1} < 0\end{aligned}$$

Per Worker Terms

- Define $x = \frac{X}{L}$ as a per worker variable. Then

$$y = \frac{Y}{L} = \frac{A\ K^{\alpha} L^{1-\alpha}}{L} = A\ \left(\frac{K}{L}\right)^{a} \left(\frac{L}{L}\right)^{1-\alpha} = A\ k^{\alpha}$$

- Per worker production function has decreasing returns to scale.

Capital Accumulation

- Capital accumulation equation: $\dot{K} = sY - \delta K$

- Important additional assumptions:

  1. Constant saving rate (very specific preferences: no r)

  2. Constant depreciation rate

- Dividing by K in the capital accumu equation: $\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta$.

- Some Algebra: $\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta = s\frac{\frac{Y}{L}}{\frac{K}{L}} - \delta = s\frac{y}{k} - \delta$

- Now remember that:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - n \Rightarrow \frac{\dot{K}}{K} = \frac{\dot{k}}{k} + n$$

- We get

$$\frac{\dot{k}}{k} + n = s\frac{y}{k} - \delta \Rightarrow \dot{k} = sy - (\delta + n)\,k$$

- Fundamental Differential Equation of Solow Model:

$$\boxed{\dot{k} = s\ A\ k^{\alpha} - (\delta + n)\,k}$$

Graphical Analysis

- Change in k, $\dot{k}$ is given by difference of $s\ A\ k^{\alpha}$ and $(\delta + n)k$

- If $s\ A\ k^{\alpha} > (\delta + n)k$, then k increases.

- If $s\ A\ k^{\alpha} < (\delta + n)k$, then k decreases.

- Steady state: a capital stock k^* where, when reached, $\dot{k} = 0$

- Unique positive steady state in Solow model.

- Positive steady state (locally) stable.

Steady State Analysis

- Steady State: $\dot{k} = 0$

- Solve for steady state

$$0 = s\ A\ (k^*)^\alpha - (n+\delta)k^* \Rightarrow k^* = \left(\frac{s\ A}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- Steady state output per worker $y^* = \left(\frac{s\ A}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

- Steady state output per worker depends positively on the saving (investment) rate and negatively on the population growth rate and depreciation rate.

Comparative Statics

- Suppose that of all a sudden saving rate s increases to $s' > s$. Suppose that at period 0 the economy was at its old steady state with saving rate s.

- $(n + \delta)k$ curve does not change.

- $s\ A\ k^{\alpha} = sy$ shifts up to $s'y$.

- New steady state has higher capital per worker and output per worker.

- Monotonic transition path from old to new steady state.

Evaluating the Basic Solow Model

- Why are some countries rich (have high per worker GDP) and others are poor (have low per worker GDP)?

- Solow model: if all countries are in their steady states, then:

    1. Rich countries have higher saving (investment) rates than poor countries

    2. Rich countries have lower population growth rates than poor countries

- Data seem to support this prediction of the Solow model

The Solow Model and Growth

- No growth in the steady state

- Positive or negative growth along the transition path:

$$\begin{aligned} \dot{k} &= s\ A\ k^{\alpha} - (n+\delta)k \\ g_k &\equiv \frac{\dot{k}}{k} = s\ A\ k^{\alpha-1} - (n+\delta) \end{aligned}$$

Introducing Technological Progress

- Aggregate production function becomes

$$Y = K^{\alpha} (AL)^{1-\alpha}$$

- A : Level of technology in period t.

- Key assumption: constant positive rate of technological progress:

$$\frac{\dot{A}}{A} = g > 0$$

- Growth is exogenous.

Balanced Growth Path

- Situation in which output per worker, capital per worker and consumption per worker grow at constant (but potentially different) rates

- Steady state is just a balanced growth path with zero growth rate

- For Solow model, in balanced growth path $g_y = g_k = g_c$

Proof

- Capital Accumulation Equation $\dot{K} = sY - \delta K$

- Dividing both sides by K yields $g_K \equiv \frac{\dot{K}}{K} = s\frac{Y}{K} - \delta$

- Remember that $g_k \equiv \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n$

- Hence

$$g_k \equiv \frac{\dot{k}}{k} = s\frac{Y}{K} - (n + \delta)$$

- In BGP g_k constant. Hence $\frac{Y}{K}$ constant. It follows that $g_Y = g_K$. Therefore $g_y = g_k$

What is the Growth Rate?

- Output per worker

$$y = \frac{Y}{L} = \frac{K^{\alpha}(AL)^{1-\alpha}}{L} = \frac{K^{\alpha}}{L^{\alpha}}\frac{(AL)^{1-\alpha}}{L^{1-\alpha}} = k^{\alpha}A^{1-\alpha}$$

- Take logs and differentiate $g_y = \alpha g_k + (1-\alpha)g_A$

- We proved $g_k = g_y$ and we use $g_A = g$ to get

$$g_k = \alpha g_k + (1-\alpha)g = g = g_y$$

- BGP growth rate equals rate of technological progress. No TP, no growth in the economy.

Analysis of Extended Model

- in BGP variables grow at rate g. Want to work with variables that are constant in long run. Define:

$$\begin{aligned}\tilde{y} &= \frac{y}{A} = \frac{Y}{AL} \\ \tilde{k} &= \frac{k}{A} = \frac{K}{AL}\end{aligned}$$

- Repeat the Solow model analysis with new variables:

$$\begin{aligned}\tilde{y} &= \tilde{k}^{\alpha} \\ \dot{\tilde{k}} &= s\tilde{y} - (n+g+\delta)\tilde{k} \\ \dot{\tilde{k}} &= s\tilde{k}^{\alpha} - (n+g+\delta)\tilde{k}\end{aligned}$$

Closed-Form Solution

- Repeating all the steps than in the basic model we get:

$$\tilde{k}(t) = \left(\frac{s}{\delta+n+g} + \left(\tilde{k}_0^{1-\alpha} - \frac{s}{\delta+n+g}\right) e^{-\lambda t}\right)^{\frac{1}{1-\alpha}}$$

$$\tilde{y}(t) = \left(\frac{s}{\delta+n+g} + \left(\tilde{k}_0^{1-\alpha} - \frac{s}{\delta+n+g}\right) e^{-\lambda t}\right)^{\frac{\alpha}{1-\alpha}}$$

- Interpretation.

Balanced Growth Path Analysis

- Solve for $\tilde{k}^*$ analytically

$$\begin{aligned} 0 &= s{\tilde{k}^*}^{\alpha} - (n+g+\delta)\tilde{k}^* \\ \tilde{k}^* &= \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

- Therefore

$$\tilde{y}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$
\begin{aligned}
k(t) &= A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \\
y(t) &= A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \\
K(t) &= L(t)A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \\
Y(t) &= L(t)A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

Evaluation of the Model: Growth Facts

1. Output and capital per worker grow at the same constant, positive rate in BGP of model. In long run model reaches BGP.

2. Capital-output ratio $\frac{K}{Y}$ constant along BGP

3. Interest rate constant in balanced growth path

4. Capital share equals α, labor share equals $1-\alpha$ in the model (always, not only along BGP)

5. Success of Solow model along these dimensions, but source of growth, technological progress, is left unexplained.

Evaluation of the Model: Development Facts

1. Differences in income levels across countries explained in the model by differences in s, n and δ.

2. Variation in growth rates: in the model *permanent* differences can only be due to differences in rate of technological progress g. *Temporary* differences are due to transition dynamics.

3. That growth rates are not constant over time for a given country can be explained by transition dynamics and/or shocks to n, s and δ.

4. Changes in relative position: in the model countries whose s moves up, relative to other countries, move up in income distribution. Reverse with n.

Interest Rates and the Capital Share

- Output produced by price-taking firms
- Hire workers L for wage w and rent capital K from households for r
- Normalization of price of output to 1.
- Real interest rate equals $r - \delta$

Profit Maximization of Firms

$$\max_{K,L} K^{\alpha} (AL)^{1-\alpha} - wL - rK$$

- First order condition with respect to capital K

$$\begin{aligned} \alpha K^{\alpha-1} (AL)^{1-\alpha} - r &= 0 \\ \alpha \left(\frac{K}{AL}\right)^{\alpha-1} &= r \\ \alpha \tilde{k}^{\alpha-1} &= r \end{aligned}$$

- In balanced growth path $\tilde{k} = \tilde{k}^*$, constant over time. Hence in BGP $r$constant over time, hence $r - \delta$ (real interest rate) constant over time.

Capital Share

- Total income $= Y$, total capital income $= rK$

- Capital share

$$\begin{aligned} \text{capital share} &= \frac{rK}{Y} \\ &= \frac{\alpha K^{\alpha-1}(AL)^{1-\alpha}K}{K^{\alpha}(AL)^{1-\alpha}} \\ &= \alpha \end{aligned}$$

- Labor share $= 1 - \alpha$.

Wages

- First order condition with respect to labor L

$$\begin{aligned}(1-\alpha)K^{\alpha}(LA)^{-\alpha}A &= w \\ (1-\alpha)\tilde{k}^{\alpha}A &= w\end{aligned}$$

- Along BGP $\tilde{k} = \tilde{k}^*$, constant over time. Since A is growing at rate g, the wage is growing at rate g along a BGP.